On the Hilbert function of one-dimensional local complete intersections [*]

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Abstract

The Hilbert function of standard graded algebras are well understood by Macaulay's theorem and very little is known in the local case, even if we assume that the local ring is a complete intersection. An extension to the power series ring R of the theory of Gröbner bases (w.r.t. local degree orderings) enable us to characterize the Hilbert function of one dimensional quadratic complete intersections $A = R/I$, and we give a structure theorem of the minimal system of generators of I in terms of the Hilbert function. We find several restrictions for the Hilbert function of A in the case that I is a complete intersection of type $(2, b)$. Conditions for the Cohen-Macaulyness of the associated graded ring of A are given.

1 Introduction and preliminaries

Let G be a standard graded K-algebra; by this we mean $G = P/I$ where $P = K[x_1, \ldots x_n]$ is a polynomial ring over the field K and I an homogeneous ideal. It is clear that for every $t \geq 0$ the set I_t of the forms of degree t in P is a K-vector space of finite dimension. For every positive integer t the Hilbert function of G is defined as follows:

$$HF_G(t) = dim_K G_t = dim_K P_t - dim_K I_t = \binom{n + t - 1}{t} - \ dim_K I_t.$$

Its generating function $HS_G(\theta) = \sum_{t \in \mathbb{N}} HF_G(t)\theta^t$ is the Hilbert Series of G.

The relevance of this notion comes from the fact that in the case I is the defining ideal of a projective variety V, the dimension, the degree and the arithmetic genus of V can be immediately computed from the Hilbert Series of P/I.

A fundamental theorem by Macaulay describes exactly those numerical functions which occur as the Hilbert functions of a standard graded K-algebra. Macaulay's Theorem says that for each t there is an upper bound for $HF_G(t + 1)$ in terms of $HF_G(t)$, and this bound is sharp in the sense that any numerical function satisfying it can be realized as the Hilbert function of a suitable homogeneous standard K-algebra. These numerical functions are called "admissible" and will be described in the next section.

It is not surprising that additional properties yield further constraints on the Hilbert function. Thus, for example, the Hilbert function of a Cohen-Macaulay standard graded algebra is completely described by another theorem of Macaulay which says that the Hilbert series admissible for a Cohen-Macaulay standard graded algebra of dimension d, are of the type

$$\frac{1 + h_1\theta + \ldots h_s\theta^s}{(1 - z)^d}$$

where $1 + h_1\theta + \ldots h_s\theta^s$ is admissible.

The Hilbert function of a local ring A with maximal ideal $\mathfrak{m}$ and residue field K is defined as follows: for every $t \geq 0$

$$HF_A(t) = dim_{\mathrm{K}} \left(\frac{\mathfrak{m}^t}{\mathfrak{m}^{t+1}} \right).$$

It is clear that $HF_A(t)$ is equal to the minimal number of generators of the ideal $\mathfrak{m}^t$ and we can see that the Hilbert function of the local ring A is the Hilbert function of the following standard graded algebra

$$gr_{\mathfrak{m}}(A) = \oplus_{t \geq 0} \mathfrak{m}^t / \mathfrak{m}^{t+1}.$$

This algebra is called the associated graded ring of the local ring $(A, \mathfrak{m})$ and corresponds to a relevant geometric construction in the case A is the localization at the origin O of the coordinate ring of an affine variety V passing through O. It turns out that $gr_{\mathfrak{m}}(A)$ is the coordinate ring of the *Tangent Cone* of V at O, which is the cone composed of all lines that are the limiting positions of secant lines to V in O.

Despite the fact that the Hilbert function of a standard graded K-algebra G is so well understood in the case G is Cohen-Macaulay, very little is known in the local case. This mainly because, passing from the local ring A to its associated graded ring, many of the properties can be lost. This is the reason why we are very far from a description of the admissible Hilbert functions for a Cohen-Macaulay local ring when $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay. We only have some small knowledge of the behavior of these numerical functions.

An example by Herzog and Waldi (see [10]) shows that the Hilbert function of a one dimensional Cohen-Macaulay local ring can be decreasing, even the number of generators of the square of the maximal ideal can be less than the number of generators of the maximal ideal itself. Further, without restrictions on the embedding dimension, the Hilbert function of a one dimensional Cohen-Macaulay local ring can present arbitrarily many "valleys" (see [5]).

Even if we restrict ourselves to the case of a complete intersection, very little is known. In [16] it has been proved that the Hilbert function of a positive dimensional codimension two complete intersection $R/(f, g)$ is non decreasing, but we have no answer to the question asked by Rossi (see [17]) whether the same is true for every one dimensional Gorenstein local ring.

In the case that the embedding dimension of the local ring is at most three, the first author gave a positive answer to a question stated by J. Sally, by proving that the Hilbert function of a one dimensional Cohen-Macaulay local ring is increasing (see [4]). But examples show that this is not true anymore if the embedding dimension is bigger than three.

All this amount of results shows that, without strong assumptions, the Hilbert function of a one-dimensional Cohen-Macaulay local ring could be very wild. This is the reason why, in this paper, we restrict ourselves to the case $A = \mathrm{K}[\![x, y, z]\!]/I$, where the ideal $I \subseteq (x, y, z)^2$ is generated by a regular sequence $\{f, g\}$ of elements of R. We will see that, even with all these strong assumptions, the problem of determining the admissible Hilbert functions is not so easy, possibly because it is strictly related to the study of curve singularities in $\mathbb{A}^3$.

If we consider the corresponding Artinian problem, then we deal with a pair of plane curves. Several papers have been written in which the Hilbert function of an Artinian complete intersection ring $A = \mathrm{K}[\![x, y]\!]/(f, g)$ has been studied in terms of the invariants of the curves $f = g = 0$ (see Iarrobino [12], Goto, Heinzer, Kim [8], Kothari [13], ...). It is an early result due to Macaulay that the Hilbert function of such a ring A verifies for every positive integer n the following inequality:

$$|HF_A(n + 1) - HF_A(n)| \leq 1.$$

It has been proved that given such a numerical function, there exists a complete intersection $I = (f, g) \subseteq \mathrm{K}[\![x, y]\!]$ with that Hilbert function, [1], [8]. Hence the problem is solved in the Artinian case and, more in general, when $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay. Conditions on the Cohen-Macaulayness of $gr_{\mathfrak{m}}(A)$ have been studied by Goto, Heinzer and Kim in [6], [7].

Classical results concerning Cohen-Macaulay local rings of dimension one will be useful in this paper. For example it is well known, see [14],[4], [18], that there exists an integer $e \geq 1$, the multiplicity of A, such that

(i) $HF_A(n) \leq e$ for all n,

(ii) If $HF_A(j) = e$ for some j, then $HF_A(n) = e$ for all $n \geq j$,

(iii) For every $j \geq 0$ we have $HF_A(j) \geq \min\{j+1, e\}$. In particular $HF_A(e-1) = e$.

The least integer r such that $HF(r) = e$ coincides with the reduction number of $\mathfrak{m}$, which is the least integer r such that $\mathfrak{m}^{r+1} = x\mathfrak{m}^r$ for some (hence any) superficial element $x \in \mathfrak{m}$. We say that the Hilbert function of A is *increasing* (resp. *strictly increasing*) if $HF(n) \leq HF_A(n+1)$ (resp. $HF(n) < HF_A(n+1)$) for all $n = 0, \cdots, r-1$.

Throughout the whole paper K denotes an algebraically closed field of characteristic zero. Let $R = \mathrm{K}[\![x_1, \ldots x_n]\!]$ be the ring of formal power series in the indeterminates $\{x_1, \cdots, x_n\}$ with coefficients in K and maximal ideal $\mathcal{M} = (x_1, \cdots, x_n)$. We denote by $\mathbb{U}(R)$ the group of units of R. Let I be an ideal of R and consider the local ring $A = R/I$ whose maximal ideal is $\mathfrak{m} := \mathcal{M}/I$.

We have seen that the Hilbert function of a local ring A is the same as that of the associated graded ring $gr_\mathfrak{m}(A)$. Hence it will be useful to recall the presentation of this standard graded algebra. For every power series $f \in R \setminus \{0\}$ we can write $f = f_v + f_{v+1} + \cdots$, where f_v is not zero and f_j is an homogeneous polynomial of degree j in P for every $j \geq v$. We say that v is the order of f, denote f_v by f^* and call it the initial form of f. If $f = 0$ we agree that its order is ∞. It is well known that $gr_\mathfrak{m}(A) = P/I^*$, where I^* is the homogeneous ideal of the polynomial ring P generated by the initial forms of the elements of I. A set of power series $f_1, \cdots, f_r \in I$ is a standard basis of I if $I^* = (f_1^*, \cdots, f_r^*)$, (see [11]). It is clear that every ideal I has a standard basis and that every standard basis is a basis. However not every basis is a standard basis. To determine a standard basis of a given ideal of R is a classical hard problem, even in the very special case we are involved with in this paper.

In order to determine the Hilbert function of such local complete intersections it seems to be hopeless to use only the theory of tangent cones. Instead we found crucial to consider the extension to the power series ring of the theory of Gröbner bases introduced by Buchberger for ideals in the polynomial ring. We can say that a mixture of the theory of enhanced standard basis with that of the ideals of initial forms has been the winning strategy for us. The use of the theory of enhanced standard bases for studying of the Hilbert function of a local ring seems to be unusual, while there are several papers in the Theory of Singularities where this topic is essential.

We recall that the notion of Gröbner basis is defined by considering a term ordering on the terms of P (i.e. a monomial ordering where all the terms are bigger than 1). Instead, we need here to consider the so called *local degree ordering,* see [9], Chapter 6, a monomial ordering on the terms of P which is not a term ordering.

We denote by $\mathbb{T}^n$ the set of terms or monomials of P; let τ be a term ordering in $\mathbb{T}^n$, and we assume that $x_1 > \cdots > x_n$. We define a new total order $\overline{\tau}$ on $\mathbb{T}^n$ in the following way: given $m_1, m_2 \in \mathbb{T}^n$ we let $m_1 >_{\overline{\tau}} m_2$ if and only if $\deg(m_1) < \deg(m_2)$ or $\deg(m_1) = \deg(m_2)$ and $m_1 >_\tau m_2$. Given $f \in R$ we denote by $\mathrm{Supp}(f)$ the support of f, i.e. if $f = \sum_{\underline{i} \in \mathbb{N}^n} a_{\underline{i}} x^{\underline{i}}$ then $\mathrm{Supp}(f)$ is the set of terms $x^{\underline{i}}$ such that $a_{\underline{i}} \neq 0$. We remark that, given f in R, there is a monomial which is the biggest of the monomials in $\mathrm{Supp}(f)$ with respect to $\overline{\tau}$: namely, since the support of f^* is a finite set, we can take the maximum with respect to τ of the elements of this set. This monomial is called the leading monomial of f with respect to $\overline{\tau}$ and is denoted by $Lt_{\overline{\tau}}(f)$. By definition we have

$$Lt_{\overline{\tau}}(f) = Lt_\tau(f^*).$$

As usual we define the leading term ideal associated to an ideal $I \subset R$ as the monomial ideal $\mathrm{Lt}_{\overline{\tau}}(I)$ generated in R by $Lt_{\overline{\tau}}(f)$ with f running in I.

In [1] a set $\{f_1, \ldots, f_r\}$ of elements of I is called an *enhanced* standard basis of I if the corresponding leading terms generate $\mathrm{Lt}_{\overline{\tau}}(I)$. Every enhanced standard basis is also a standard basis, but the converse is not true. In [9] an enhanced standard basis of I is simply called a standard basis. We have $\mathrm{Lt}_{\overline{\tau}}(I)P = Lt_\tau(I^*)$ (see [1] Proposition 1.5.) so that

$$HF_{R/I} = HF_{P/I^*} = HF_{R/\mathrm{Lt}_{\overline{\tau}}(I)}.$$

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In the Theory of enhanced standard basis a crucial result is the Grauert's Division theorem, [9, Theorem 6.4.1]. It claims the following. Given a set of formal power series $f, f_1, \cdots, f_m \in R$ there exist power series $q_1, \ldots, q_m, r \in R$ such that $f = \sum_{j=1}^{m} q_j f_j + r$ and, for all $j = 1, \ldots, m$,

(1) No monomial of r is divisible by $Lt_{\overline{\tau}}(f_j)$,

(2) $Lt_{\overline{\tau}}(q_j f_j) \leq Lt_{\overline{\tau}}(f)$ if $q_j \neq 0$.

With the above result we can define

$$NF(f|\{f_1, \ldots, f_m\}) := r$$

and obtain in this way a reduced normal form of any power series f with respect to a given finite subset of R. The existence of a reduced normal form is the basis to obtain, in the formal power series ring, all the properties of Gröbner basis already proved in the classical case. In particular Buchberger's criterion holds for the power series ring $K[\![x_1, \ldots x_n]\!]$, see [9, Theorem 1.7.3]. A similar approach was introduced by Mora in 1982 in the localization of P, (see [15]).

We come now to describe the content of the paper. The main result is the description of all the numerical functions which are the Hilbert functions of what we call a quadratic complete intersection of codimension two in $K[\![x, y, z]\!]$. By this we mean local rings of type $K[\![x, y, z]\!]/(f, g)$ where f and g are power series of order two which form a regular sequence in $K[\![x, y, z]\!]$ with the property that $g^* \notin (f^*)$.

We first prove in Proposition 2.2 that for the Hilbert function of such local rings with a given multiplicity e, there are only two possibilities:

(1) either is increasing by one up to reach the multiplicity, say

$$\{1, 3, 4, 5, 6, 7, ..., e - 1, e, e, e....\},$$

(2) or it is increasing by one with a flat in position n which is unique, by which we mean that for some $n \leq e - 3$ we have the sequence

0	1	2	3	4	...	n-1	n	n+1	n+2	...	e-2	e-1	e	...
1	3	4	5	6	...	n+1	n+2	n+2	n+3	...	e-1	e	e	...

It turns out that if the Hilbert function is increasing by one, case (1), there is no constriction on the multiplicity. Instead, if the Hilbert function has a flat, case (2), the multiplicity e cannot be too big, namely we must have $e \leq 2n$. This unexpected result is proved in Theorem 3.6 which is the main result of this paper. Examples 2.3 and 3.7 show that the above Hilbert functions are realizable.

We present also two more results on the Cohen-Macaulayness of the tangent cone of such complete intersections. First, in Proposition 2.5, we prove that a quadratic complete intersection of codimension two in $K[\![x, y, z]\!]$ with Hilbert function increasing by one has an associated graded ring which is Cohen-Macaulay. Finally, as a second application of the methods we used in the proof of the main theorem, we are able to prove in Proposition 3.8 that for a quadratic complete intersection $A = K[\![x, y, z]\!]/I$, the tangent cone is Cohen-Macaulay in the case the vector space I^*_2 does not contain a square of a linear form.

Section four is devoted to give a structure theorem, modulo analytic isomorphisms, of the minimal system of generators of quadratic complete intersection ideals I of codimension two in $K[\![x, y, z]\!]$, Theorem 4.1 and Theorem 4.2. These results are obtained by taking into account the two possible Hilbert functions that can occur for such an ideal.

In the last section of the paper we give several examples to illustrate our results, as well as possible extensions.

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2   Ideals of type $(2, b)$

From now on we assume that $A = \mathrm{K}[\![x, y, z]\!]/I$ where I is a codimension two complete intersection ideal of $R = \mathrm{K}[\![x, y, z]\!]$. Given the integers $b \geq a \geq 2$, we say that A is *of type* (a, b), or I is of type (a, b), if I can be generated by a regular sequence $\{f, g\}$ such that order(f)=a, order(g)=b and $g^* \notin (f^*)$. In the language of [11, Chapter III, Section 1] we write $\nu^*(I) = (a, b)$ with the meaning that I is of type (a, b).

In this paper we will be mainly concerned with local rings of type $(2, 2)$; however in this section properties of local rings of type $(2, b)$ will be considered.

In the following Proposition we prove that the Hilbert function of a local ring of type $(2, b)$ verifies for every $n \geq 1$ the inequalities

$$0 \leq HF_A(n + 1) - HF_A(n) \leq 1. \tag{1}$$

The question now is whether every numerical function H such that $H(0) = 1$, $H(1) = 3$ and verifies (1) is the Hilbert function of some local ring of type $(2, b)$. This is not the case because, for example, the numerical function $\{1, 3, 4, 5, 5, 6, 7, 7, ...\}$ verifies (1) but we will see later that it cannot be the Hilbert function of a local ring of type $(2, b)$.

A local ring A of type $(2, b)$ is Cohen-Macaulay of embedding dimension three so that we know that the Hilbert function is not decreasing. We say that HF_A *admits a flat in position* n if

$$HF_A(n) = HF_A(n + 1) < e.$$

The first basic properties of the Hilbert function of a local ring of type $(2, b)$ are collected in the following proposition which is an easy consequence of the classical Macaulay Theorem.

We recall that given two positive integer n and c, the n-binomial expansion of c is

$$c = \binom{c_n}{n} + \binom{c_{n-1}}{n-1} + \cdots \binom{c_j}{j}$$

where $c_n > c_{n-1} > \cdots c_j \geq j \geq 1$. We let

$$c^{<n>} = \binom{c_n + 1}{n + 1} + \binom{c_{n-1} + 1}{n} + \cdots \binom{c_j + 1}{j + 1}.$$

The Theorem of Macaulay states that a numerical function $\{h_0, h_1, \cdots, h_i, \cdots, \}$ is the Hilbert function of a standard graded algebra if and only if $h_0 = 1$ and $h_{i+1} \leq h_i^{<i>}$ for every $i \geq 1$. We remark that if $n + 1 \leq c \leq 2n$ then the n-binomial expansion of c is

$$c = \binom{n + 1}{n} + \binom{n - 1}{n - 1} + \cdots \binom{2n - c + 1}{2n - c + 1},$$

so that $c^{<n>} = c + 1$.

Further, if $f_1, \ldots, f_r$ are elements of order $d_1, \ldots, d_r$ in the regular local ring $(R, \mathcal{M})$ and J the ideal they generate, it is known that

$$J_n^* = (J \cap \mathcal{M}^n + \mathcal{M}^{n+1})/\mathcal{M}^{n+1}$$

and

$$(f_1^*, \ldots, f_r^*)_n = (\sum_{i=1}^{r} \mathcal{M}^{n-d_i} f_i + \mathcal{M}^{n+1})/\mathcal{M}^{n+1}$$

for every non negative integer n. With this notation we have the following basic lemma.

Lemma 2.1. *Let $I = (f, g)$ be an ideal of $(R, \mathcal{M})$ with order$(f) = 2 \leq$ order$(g) = b$. Then*

(i) $I^*_j = (f^*)_j$ *for every integer $2 \leq j < b$.*

(ii) $I^*_b = (f^*, g^*)_b$.

(iii) If $g^* \notin (f^*)$ then $I^*_{b+1} = (f^*, g^*)_{b+1}$.

Proof. Since $j + 1 \leq b$ we have $g \in \mathcal{M}^b \subseteq \mathcal{M}^{j+1} \subseteq \mathcal{M}^j$, hence

$$(f, g) \cap \mathcal{M}^j + \mathcal{M}^{j+1} = (g) + (f) \cap \mathcal{M}^j + \mathcal{M}^{j+1} = f\mathcal{M}^{j-2} + \mathcal{M}^{j+1}.$$

The first assertion follows. We prove now (ii). We have:

$$(f, g) \cap \mathcal{M}^b = (g) + (f) \cap \mathcal{M}^b = (g) + f\mathcal{M}^{b-2}.$$

As for (iii) we need to prove that if $g^* \notin (f^*)$ then $(f, g) \cap \mathcal{M}^{b+1} = f\mathcal{M}^{b-1} + g\mathcal{M}$. The inclusion $\supseteq$ is clear, so let $\alpha = cf + dg \in \mathcal{M}^{b+1}$. If $d \in \mathcal{M}$ then $cf \in \mathcal{M}^{b+1}$ and this implies $c \in \mathcal{M}^{b-1}$ as required. If $d \notin \mathcal{M}$ then $g \in ((f) + \mathcal{M}^{b+1}) \cap \mathcal{M}^b = \mathcal{M}^{b+1} + f\mathcal{M}^{b-2}$ which implies $g^* \in (f^*)$, a contradiction. $\square$

Proposition 2.2. *Let $A = R/I$ be a local ring of type $(2, b)$ and $I = (f, g)$ with $\mathrm{order}(f) = 2$, $\mathrm{order}(g) = b$ and $g^* \notin (f^*)$. Then the following properties hold.*

(i) $HF_A(j) = 2j + 1$ if $j < b$.

(ii) $HF_A(b) = 2b$.

(iii) $HF_A(j - 1) \leq HF_A(j) \leq HF_A(j - 1) + 1$ if $j \geq b$.

(iv) HF_A admits at most $b - 1$ flats.

Proof. By (i) of the above Lemma we have for every $j < b$

$$HF_A(j) = HF_{P/I^*}(j) = HF_{P/(f^*)}(j) = 2j + 1.$$

We prove now the second assertion. By (ii) of the above Lemma we have

$$HF_A(b) = HF_{P/I^*}(b) = HF_{P/(f^*, g^*)}(b).$$

Since $g^* \notin (f^*)$ we get $HF_A(b) = HF_{P/(f^*)}(b) - 1 = 2b + 1 - 1 = 2b$ as required.

As for (iii) we need only to prove that $HF_A(j) \leq HF_A(j - 1) + 1$ if $j \geq b$. We have $HF_A(b) = 2b$, $HF_A(b - 1) = 2b - 1$, hence we can argue by induction on j. Let $j \geq b$ and assuming $HF_A(j) \leq HF_A(j - 1) + 1$ we need to prove that $HF_A(j + 1) \leq HF_A(j) + 1$.

We have $j + 1 \leq HF_A(j) < HF_{P/(f^*)}(j) = 2j + 1$, hence, by the remark before the Lemma, we get

$$HF_A(j + 1) \leq HF_A(j)^{<j>} = HF_A(j) + 1$$

as wanted.

Finally we prove (iv). We have $HF_A(b) = 2b$ and at each step HF_A goes up at most by one. Hence, if there are p flats between b and j, we have $HF_A(j) = 2b + j - b - p$. But $HF_A(j) \geq j + 1$, so that $p \leq b - 1$. $\square$

From the above proposition it follows that the Hilbert function of a local ring of type $(2, b)$ either is strictly increasing or it has one or more flats (no more than $b - 1$); if the first is the case, it has the following shape

$$HF_A(j) = \begin{cases} 2j + 1 & j = 0, \ldots, b - 1, \\ j + b & b \leq j \leq e - b, \\ e & j \geq e - b + 1. \end{cases} \tag{2}$$

where e and b are integers, $b \geq 2$ and $e \geq 2b$.

We show with the following example that given a numerical function H as in (2) we can find a local ring of type $(2, b)$ with multiplicity e whose Hilbert function is H.

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Example 2.3. Let $b \geq 2$ and $e \geq 2b$. We claim that the above numerical function is the Hilbert function of the following local ring of type $(2, b)$ and multiplicity e.

Let $I = (x^2 + y^{e-2b+2}, xy^{b-1})$ and $A = K[\![x, y, z]\!]/I$. We fix an ordering on the monomials of P with the property that $x > y$. We let $f := x^2 + y^{e-2b+2}$, $g := xy^{b-1}$ and claim that $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy^{b-1}, y^{e-b+1})$.

Since $e \geq 2b$ and $x > y$ it is clear that $\mathrm{Lt}_{\overline{\tau}}(f) = x^2$. We have

$$S(f, g) = y^{b-1}f - xg = y^{b-1}(x^2 + y^{e-2b+2}) - xxy^{b-1} = y^{e-b+1}.$$

Let $h := S(f, g) = y^{e-b+1}$, then

$$S(f, h) = y^{e-b+1}f - x^2h = y^{e-b+1}(x^2 + y^{e-2b+2}) - x^2y^{e-b+1} = y^{2e-3b+3} = y^{e-2b+2}h$$

and $S(g, h) = 0$. It follows that

$$
\begin{aligned}
\mathrm{NF}(S(f, g) \,|\{h\}) &= \mathrm{NF}(h \,|\{h\}) = 0, \\
\mathrm{NF}(S(f, h) \,|\{h\}) &= \mathrm{NF}(y^{e-2b+2}h \,|\{h\}) = 0, \\
\mathrm{NF}(S(g, h) \,|\{h\}) &= \mathrm{NF}(0 \,|\{h\}) = 0.
\end{aligned}
$$

By Buchberger criterion we get that $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy^{b-1}, y^{e-b+1})$ as claimed. With the aim of a simple computation we can prove that $K[x, y, z]/(x^2, xy^{b-1}, y^{e-b+1})$ has the above Hilbert function; clearly the same is true for the local ring $K[\![x, y, z]\!]/(x^2 + y^{e-2b+2}, xy^{b-1})$.

We end this section by proving that for a local ring of type $(2, b)$ the condition that the Hilbert function is strictly increasing is equivalent to the Cohen-Macaulayness of the tangent cone. First we need to prove that the property of having type (a, b) can be carried on the quotient modulo a suitable superficial element. We recall that an element $\ell \in \mathcal{M}$ is superficial for $\mathcal{M}/I$ if ℓ does not belong to any of the associated primes of I^* different from the homogeneous maximal ideal. Since the residual field if infinite the existence of superficial elements is guaranteed. Moreover, it is easy to prove:

Proposition 2.4. *Let I be an ideal of R of type (a, b) with $2 \leq a \leq b$. There exists $\ell \in \mathcal{M} \setminus \mathcal{M}^2$ such that*

  (i) the coset of ℓ in R/I is superficial for $\mathcal{M}/I$,

  (ii) $\overline{I} = I + (\ell)/(\ell)$ is an ideal of $R/(\ell)$ of type (a, b).

Proof. It is well known that ℓ verifies (i) if ℓ^* does not belong to any of the associated prime ideals of I^* (different from the homogeneous maximal ideal). Let $I = (f, g)$ be with $order(f) = a \leq order(g) = b$. Then it is easy to see that $\overline{I}$ satisfies (ii) provided:
  a) ℓ^* does not divide f^*
  b) $g^* \notin (f^*, \ell^*)$.
In fact $\overline{I} = (\overline{f}, \overline{g})$ in R/ℓ and condition a) assures $order(\overline{f}) = a$ and condition b) gives $\overline{g}^* \notin (\overline{f}^*)$. Since $depth\, P/(f^*, g^*) \geq 1$ $(P = \mathrm{K}[x, y, z])$, it is easy to see that for having a) and b) it is enough to choose $\ell \in \mathcal{M} \setminus \mathcal{M}^2$ such that ℓ^* is regular in $P/(f^*, g^*)$. Clearly, if this is the case, ℓ^* does not divide f^* and if $g^* \in (f^*, \ell^*)$, then $g^* = \alpha f^* + \beta \ell^*$ with $\alpha, \beta \in P$. Since ℓ^* is $P/(f^*, g^*)$-regular, then $\beta \in (f^*, g^*)$. Hence $g^* = \alpha f^* + \ell^*(\beta_1 g^* + \beta_2 \ell^*)$, so $g^*(1 - \ell\beta_1) \in (f^*)$, a contradiction because $g^* \notin (f^*)$. Since the residue field is infinite, an element $\ell \in \mathcal{M} \setminus \mathcal{M}^2$ verifying the conditions of the proposition can be selected by avoiding the associated prime ideals to I^* and to (f^*, g^*). $\square$

It is well known that if the associated graded ring $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay, then the Hilbert function of A is strictly increasing. However the converse is in general very rare. In the following result we will show a special case where this implication holds true.

Proposition 2.5. *Let $A = R/I$ be a local ring of type $(2, b)$. Then $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay if and only if HF_A is strictly increasing.*

Proof. Let $I = (f,g)$ with order$(f) = 2$, order$(g) = b$ and $g^* \notin (f^*)$. If the associated graded ring is Cohen-Macaulay, then its Hilbert function is strictly increasing and thus the Hilbert function of A is strictly increasing as well. Assume that HF_A is strictly increasing. From Proposition 2.2, a simple computation gives

$$\Delta HF_A(n) := HF_A(n+1) - HF_A(n) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \dots, b-1, \\ 1 & n = b, \dots, r-1, \\ 0 & n \geq r \end{cases}$$

with $r = e - b + 1$.

From Proposition 2.4 there exists a superficial element $x \in A$ such that

$$HF_{A/xA}(n) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \dots, b-1, \\ 1 & n = b. \end{cases}$$

From Macaulay's characterization of Hilbert functions and the fact that $e(A/xA) = e(A)$, we get $\Delta HF_A = HF_{A/xA}$. Hence $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay, [19]. $\square$

Notice that the above proposition cannot be extended to local rings of type (a, b) with $a > 2$, as the following example shows. Consider the local ring $A = R/I$ where $I = (x^4, x^2y + z^4) \subseteq R = \mathrm{K}[\![x, y, z]\!]$; A is a one-dimensional Gorenstein local ring and

$$HF_A = \{1, 3, 6, 9, 11, 13, 14, 15, 16, 16, \dots, \}$$

is strictly increasing. Now it is clear that $x^4, x^2y \in I^*$ and since $x^2(x^2y + z^4) - yx^4 \in I$, also $x^2z^4 \in I^*$. This implies that $x^3z^3(x, y, z) \subseteq I^*$; since $x^3z^3 \notin I^*$ $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay.

A natural and general problem would be to characterize the Hilbert functions of all the ideals I of type $(2, b)$. If the Hilbert function has one or more flat, the behavior is difficult to control. However if we denote by p the number of flats, by Proposition 2.2 we know that $p \leq b - 1$. With the aid of huge computations made with CoCoa, we ask the following question.

Question 2.6. *Let $A = R/I$ be a local ring of type $(2, b)$ with $b \geq 2$ and multiplicity e. Let $n := \min\{j : HF_{R/I}(j) = HF_{R/I}(j+1) < e\}$ and let p be the number of flats. Then*

$$e \leq (p+1)n \ (\leq bn).$$

The main result of the paper answer the question in the case $a = b = 2$.

3 The main result

In this section we present a complete characterization of the numerical functions which are the Hilbert functions of local rings of type $(2, 2)$. In particular we prove that certain monomial ideals cannot be the initial ideals of a complete intersection, a relevant task even in the graded setting (see for example [3]).

By the definition we gave in the above section, a local ring $A = \mathrm{K}[\![x, y, z]\!]/I$ of type $(2, 2)$ is of dimension one, $HF_A(1) = 3$ and $HF_A(2) = 4$. In particular I can be generated by a regular sequence, say $I = (f, g)$, where f and g are power series of order two such that f^* and g^* are linearly independent in the vector space $K[x, y, z]_2$. We recall that by Proposition 2.2 we have $I^*_2 = < f^*, g^* >$ and $I^*_3 = (f^*, g^*)_3 = < f^*x, f^*y, f^*z, g^*x, g^*y, g^*z >$.

Since $HF_A(2) = 4$ we know that $4 \leq HF_A(3) \leq HF_A(2)^{<2>} = 5$. If $HF_A(3) = 4$, then $6 = dim(I^*_3) = \dim_{\mathrm{K}} < f^*x, f^*y, f^*z, g^*x, g^*y, g^*z >$. This easily implies that f^* and g^* form a regular

sequence in $K[x, y, z]$. As a consequence $I^* = (f^*, g^*)$ and the Hilbert function of A is $\{1, 3, 4, 4, 4, ...\}$ which is as in (2) with $b = 2$, $e = 4$.

We want to study the remaining case, when $HF_A(3) = 5$. We first remark that in this case f^* and g^* share a common factor, say L, which must be a linear form because f^* and g^* are linearly independent. Hence we can write

$$f^* = LM, \quad g^* = LN$$

where L, M, N are linear forms in $K[x, y, z]$ such that M and N are linearly independent. In particular $I^*_2 = < LM, LN >$.

We have two possibilities, either L, M, N are linearly independent or are linearly dependent. We remark that this property depends on the ideal I and not on the generators of I. Namely, if we say that I^*_2 is *square free* with the meaning that it does not contain a square of a linear form, we can prove the following easy result:

Lemma 3.1. *With the above assumption, the vectors L, M, N are linearly independent if and only if I^*_2 is square-free.*

Proof. Let us first assume that L, M, N are linearly dependent. Since M, N are linearly independent we have $L = \alpha M + \beta N$ so that $L^2 = \alpha LM + \beta LN \in < I^*_2 >$. Hence I^*_2 is not square-free.

We prove now that if L, M, N are linearly independent then I^*_2 is square-free. Let P be a linear form such that $P^2 \in I^*_2 = < LM, LN >$; then $P \in (L)$ so that $P = \lambda L$. We have $\lambda^2 L^2 = \alpha LM + \beta LN$ hence $\lambda^2 L = \alpha M + \beta N$; since L, M, N are linearly independent this implies $\lambda = 0$ and finally $P = 0$. $\square$

For completeness, we need now to recall the notion of k-algebra isomorphism. Given a set of minimal generators $\underline{y} = \{y_1, y_2, ..., y_n\}$ of the maximal ideal of $R = K[\![x_1, \ldots x_n]\!]$, we let $\phi_{\underline{y}}$ be the automorphism of R which is the result of substituting y_i for x_i in a power series $f(x_1, x_2, ..., x_n) \in R$. Given two ideals I and J in R it is well known that there exist a K-algebra isomorphism $\alpha : R/I \to R/J$ if and only if for some generators $y_1, y_2, ..., y_n$ of the maximal ideal of R, we have $I = \phi_{\underline{y}}(J)$.

We start now by deforming, up to isomorphism, the generators f and g of the given ideal I.

Lemma 3.2. *Let $A = R/I$ be a local ring of type $(2, 2)$ such that $HF_A(3) = 5$.*

(i) If I^*_2 is not square-free we may assume, up to isomorphism, that $I = (f, g)$ with $f^* = x^2$ and $g^* = xy$.

(ii) If I^*_2 is square-free we may assume, up to isomorphism, that $I = (f, g)$ with $f^* = xy$ and $g^* = xz$,

Proof. Let us first assume that I^*_2 is not square-free; then $f^* = LM$, $g^* = LN$ with L, M, N linearly dependent; since M and N are linearly independent, we must have $L = \lambda M + \rho N$ for suitable λ and ρ in K with $(\lambda, \rho) \neq (0, 0)$. By symmetry we may assume $\lambda \neq 0$. Then it is easy to see that L and N are linearly independent so that we can consider an automorphism ϕ sending $x \to L, y \to N$. We have $f = LM + a$ and $g = LN + b$ for suitable $a, b \in \mathcal{M}^3$, and further

$$L^2 = \lambda LM + \rho LN = \lambda f + \rho g - \lambda a - \rho b.$$

We get

$$I = (f, g) = (\lambda f, g) = (L^2 - \rho g + \lambda a + \rho b, g) = (L^2 + \lambda a + \rho b, g) =$$
$$= (L^2 + \lambda a + \rho b, LN + b) = \phi((x^2 + \phi^{-1}(\rho b + \lambda a), xy + \phi^{-1}(b)).$$

The conclusion follows.

Now we assume that I^*_2 is square-free. Then $f^* = LM$ and $g^* = LN$ where L, M, N are linear forms in $K[x, y, z]$ which are linearly independent. As before we have $f = LM + a$ and $g = LN + b$ for suitable $a, b \in \mathcal{M}^3$,

Let us consider the automorphism ϕ sending $x \to L, y \to M, z \to N$. We have

$$I = (f, g) = (LM + a, LN + b) = \phi((xy + \phi^{-1}(a), xz + \phi^{-1}(b)))$$

and the conclusion follows. $\square$

Using Grauert division theorem, we can prove a first useful preparation result in the case $x^2 = \mathrm{Lt}_{\overline{\tau}}(f)$ and $xy = \mathrm{Lt}_{\overline{\tau}}(g)$.

Lemma 3.3. *Let $A = R/I$ be a local ring of type $(2,2)$ such that $I = (f,g)$, $\mathrm{Lt}_{\overline{\tau}}(f) = x^2$, $\mathrm{Lt}_{\overline{\tau}}(g) = xy$. Then we can write*

$$I = (x^2 + axz^p + F(y,z), xy + bxz^q + G(y,z))$$

where $p,q \geq 1$, $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $b = 0$ or $b \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $F, G \in \mathrm{K}[\![y,z]\!]_{\geq 2}$.

Proof. By the assumption we have $f = x^2 + F$ with $\mathrm{Lt}_{\overline{\tau}}(F) <_{\overline{\tau}} x^2$ and $g = xy + G$ with $\mathrm{Lt}_{\overline{\tau}}(G) <_{\overline{\tau}} xy$. Applying Grauert's division theorem to the power series F, f, g we get

$$F = \alpha f + \beta g + r$$

where $\alpha, \beta, r \in R$, no monomial of $\mathrm{Supp}(r)$ is divisible by x^2 or xy, and

$$\mathrm{Lt}_{\overline{\tau}}(\alpha f), \mathrm{Lt}_{\overline{\tau}}(\beta g) \leq_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(F) <_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(f) = x^2.$$

We can write $\alpha = \sum_{i \geq 0} \alpha_i$, where, for every i, α_i is a degree i form in $\mathrm{K}[\![x,y,z]\!]$. It is clear that the initial form of $\alpha f = \alpha(x^2 + F)$ is $\alpha_0 x^2 + \alpha_0 F_2$ so that $\alpha_0 = 0$, otherwise $\mathrm{Lt}_{\overline{\tau}}(\alpha f) = x^2$. In particular $1 - \alpha$ is a unit. Since

$$(1 - \alpha)f = f - \alpha f = x^2 + F - \alpha f = x^2 + r + \beta g$$

we get

$$I = (f,g) = (x^2 + r, g).$$

We apply now Grauert's Division Theorem to the power series $G, x^2 + r, f$ where $G = g - xy$ and $\mathrm{Lt}_{\overline{\tau}}(G) <_{\overline{\tau}} xy$. We get

$$g - xy = G = t(x^2 + r) + sg + r'$$

where no monomial of $\mathrm{Supp}(r')$ is divisible by $\mathrm{Lt}_{\overline{\tau}}(x^2 + r) = \mathrm{Lt}_{\overline{\tau}}(x^2 + F - \alpha f - \beta g) = x^2$ or by $\mathrm{Lt}_{\overline{\tau}}(g) = xy$.

Since $g = xy + t(x^2 + r) + sg + r'$, we get

$$g(1 - s) = t(x^2 + r) + r' + xy$$

and we claim that $1 - s$ is a unit. Namely, $\mathrm{Lt}_{\overline{\tau}}(sg) \leq \mathrm{Lt}_{\overline{\tau}}(G) < xy$ and, as before,

$$sg = s(xy + G) = s_0(xy + G) + s_1(xy + G) + ...$$

This implies $s_0 = 0$, otherwise $\mathrm{Lt}_{\overline{\tau}}(sg) = xy$. This proves the claim.

Now we have

$$I = (x^2 + r, g) = (x^2 + r, (1 - s)g) = (x^2 + r, t(x^2 + r) + r' + xy) = (x^2 + r, xy + r'),$$

where no monomial of $\mathrm{Supp}(r)$ and $\mathrm{Supp}(r')$ is divisible by x^2 or xy.

It is easy to see that this implies

$$\begin{aligned} r &= axz^p + F(y,z) \\ r' &= bxz^q + G(y,z) \end{aligned}$$

with $p, q \geq 1$, $F, G \in \mathrm{K}[\![y,z]\!]_{\geq 2}$, $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$, and $b = 0$ or $b \in \mathbb{U}(\mathrm{K}[\![z]\!])$. $\qquad\square$

We can prove now the main preparation result.

Theorem 3.4. *Let $A = R/I$ be a local ring of type $(2,2)$ such that $HF_A(3) = 5$.*

a) If I_2^* is not square-free then, up to isomorphism, we can write

$$I = (x^2 + axz^p + F(y,z), xy + G(y,z))$$

where $p \geq 2$, $a \in \{0,1\}$, and $F, G \in K[\![y,z]\!]_{\geq 3}$.

b) If I_2^* is square-free then, up to isomorphism, we can write

$$I = (x^2 + xz + F(y,z), xy + dyz + \alpha y^r + \beta z^s)$$

where $F \in K[\![y,z]\!]_{\geq 3}$, $d \in K[\![y,z]\!]$, $d(0,0) = 1$, $r, s \geq 3$, $\alpha = 0$ or $\alpha \in U(K[\![y]\!])$, $\beta = 0$ or $\beta \in U(K[\![z]\!])$.

Proof. By Lemma 3.2, up to isomorphism, we can find generators f and g of I such that either $f^* = x^2$ and $g^* = xy$ or $f^* = xy$ and $g^* = xz$.

Let us first assume that $f^* = x^2$ and $g^* = xy$; then

$$Lt_{\overline{\tau}}(f) = Lt_\tau(f^*) = Lt_\tau(x^2) = x^2$$

$$Lt_{\overline{\tau}}(g) = Lt_\tau(g^*) = Lt_\tau(xy) = xy,$$

so that, as remarked at the end of the proof of Lemma 3.3, we have $I = (x^2 + r, xy + r')$ where no monomial of $\mathrm{Supp}(r)$ and $\mathrm{Supp}(r')$ is divisible by x^2 or xy.

Since $f^* = x^2$ and $g^* = xy$, we also have $f = x^2 + h$, $g = xy + s$ where $\mathrm{order}(h), \mathrm{order}(s) \geq 3$. This implies $I = (x^2 + r, xy + r') = (x^2 + h, xy + s)$ and $I_2^* = <x^2, xy>$, the vector space spanned by x^2 and xy.. Since the degree 2 component of r is a linear combination of the monomials xz, y^2, yz, z^2, it must be zero, otherwise the leading form of $x^2 + r$ cannot be in $I_2^* = <x^2, xy>$. This proves that the order of r is at least 3. Exactly in the same way we can prove that this holds true also for r'.

It is easy to see that this implies

$$r = axz^p + D(y,z), \qquad r' = bxz^q + E(y,z)$$

where $p, q \geq 2$, $a = 0$ or $a \in U(K[\![z]\!])$, $b = 0$ or $b \in U(K[\![z]\!])$, and $D, E \in K[\![y,z]\!]_{\geq 3}$.

Now let ϕ be the automorphism of $K[\![x,y,z]\!]$ defined by

$$x \to x, \quad y \to y - bz^q, \quad z \to z$$

and let $S := \phi(D)$ and $T := \phi(E)$. Then $S, T \in K[\![y,z]\!]_{\geq 3}$ and we have

$$\phi(f) = \phi(x^2 + r) = \phi(x^2 + axz^p + D(y,z)) = x^2 + axz^p + S(y,z))$$

and

$$\phi(g) = \phi(xy + r') = \phi(xy + bxz^q + E(y,z)) = x(y - bz^q) + bxz^q + \phi(E(y,z)) = xy + T(y,z)).$$

This implies that, up to isomorphism, we may assume

$$I = (x^2 + axz^p + S(y,z), xy + T(y,z))$$

with $p \geq 2$, $a = 0$ or $a \in U(K[\![z]\!])$, $b = 0$ or $b \in U(K[\![z]\!])$, and $S, T \in K[\![y,z]\!]_{\geq 3}$.

Now if $a = 0$ we are done, otherwise let $a \neq 0$. Since the ground field K is algebraically closed and a is invertible in $K[\![z]\!]$, a straightforward application of Hensel Lemma enables us to find an element $c \in R$ such that $c^p = a$.

Let us consider the automorphism $\phi : K[\![x,y,z]\!] \to K[\![x,y,z]\!]$ defined by

$$x \to x, \quad y \to y, \quad z \to cz.$$

If F and G are power series in $K[[z]]$ such $\phi(F) = S$ and $\phi(G) = T$, then

$$\phi(x^2 + xz^p + F) = x^2 + xc^p z^p + S = x^2 + axz^p + S, \qquad \phi(xy + G) = xy + T.$$

The conclusion easily follows.

We need now to consider the other case when $f^* = xy$, $g^* = xz$. As before we choose a monomial order τ such that $x >_\tau z$ and let ϕ be the automorphism of $K[\![x, y, z]\!]$ defined by

$$x \to x + z, \quad y \to x, \quad z \to y.$$

We have $f = xy + d$, $g = xz + e$ where d and e have order at least 3. Hence

$$\phi(f) = (x + z)x + \phi(d) = x^2 + xz + h, \quad \phi(g) = (x + z)y + \phi(e) = xy + yz + s$$

where $h := \phi(d)$ and $s := \phi(e)$ have order ≥ 3. Thus, up to isomorphism, we may assume that I is generated by the power series $x^2 + xz + h$ and $xy + yz + s$; this implies that $I_2^* =< x^2 + xz, xy + yz >$.

Since $x^2 >_\tau xz$ and $xy >_\tau yz$, we get

$$\mathrm{Lt}_{\overline{\tau}}(x^2 + xz + h) = Lt_\tau((x^2 + xz + h)^*) = Lt_\tau(x^2 + xz) = x^2$$

$$\mathrm{Lt}_{\overline{\tau}}(xy + yz + s) = Lt_\tau((xy + yz + s)^*) = Lt_\tau(xy + yz) = xy$$

and we may use Lemma 3.3 to get

$$I = (x^2 + axz^p + S(y, z), xy + bxz^q + M(y, z))$$

where $p, q \geq 1$, $a = 0$ or $a \in \mathbb{U}(K[\![z]\!])$, $b = 0$ or $b \in \mathbb{U}(K[\![z]\!])$, $S, M \in K[\![y, z]\!]_{\geq 2}$.

Now let $\alpha := x^2 + axz^p + S(y, z)$; if $p \geq 2$ then $\alpha^* = x^2 + S(y, z)_2$ is an element of the vector space $I_2^* =< x^2 + xz, xy + yz >$, a contradiction. Hence $p = 1$ and thus we get $\alpha^* = x^2 + a_0 xz + S(y, z)_2 \in< x^2 + xz, xy + yz >$. This clearly implies $a_0 = 1$ and $S(y, z)_2 = 0$, so that the order of $S(y, z)$ is at least 3.

Now let $\beta := xy + bxz^q + M(y, z)$; if $b \neq 0$ and $q = 1$ then $b_0 \neq 0$ and we have

$$\beta^* = xy + b_0 xz + M(y, z)_2 \in I_2^* =< x^2 + xz, xy + yz >,$$

a contradiction. Hence it must be either $b = 0$ or $q \geq 2$; in both cases we have

$$\beta^* = xy + M(y, z)_2 \in< x^2 + xz, xy + yz >$$

which implies $M(y, z) = yz + H(y, z)$ where $H(y, z)$ is a power series in $K[[y, z]]$ with order at least 3.

At this point we have $I = (x^2 + axz + S(y, z), xy + bxz^q + yz + H(y, z))$ with $a_0 = 1$, S and H $\in K[[y, z]]_{\geq 3}$ and either $b = 0$ or $q \geq 2$.

Let us consider the automorphism ϕ given by

$$x \to x, \quad y \to y - bz^q, \quad z \to z.$$

We get

$$\phi(x^2 + axz + S(y, z)) = x^2 + axz + B(y, z),$$

and

$$\phi(xy + bxz^q + yz + H(y, z)) = x(y - bz^q) + bxz^q + (y - bz^q)z + \phi(H) = xy + yz + L(y, z)$$

where $B(y, z) := \phi(S)$ and $L(y, z) := -bz^{q+1} + \phi(H) \in K[[y, z]]_{\geq 3}$.

Hence, up to isomorphism, we may assume $I = (x^2 + axz + B(y, z), xy + yz + L(y, z))$ with $a_0 = 1$ and $B, L \in K[[y, z]]_{\geq 3}$. Now it is clear that since $L(y, z)$ has order at least 3, we can write $L(y, z) = cyz + \alpha y^r + \beta z^s$ with $c \in K[y, z]_{\geq 1}$, $r, s \geq 3$, $\alpha = 0$ or $\alpha \in \mathbb{U}(K[\![y]\!])$, and $\beta = 0$ or $\beta \in \mathbb{U}(K[\![z]\!])$. Hence

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we get $I = (x^2 + axz + B(y,z), xy + yz + cyz + \alpha y^r + \beta z^s)$. We let $d := 1 + c$ so that $d \in K[[y,z]]$, $d(0,0) = 1 + c(0,0) = 1$ and

$$I = (x^2 + axz + B(y,z), xy + dyz + \alpha y^r + \beta z^s).$$

Finally let us consider the automorphism ϕ given by

$$x \to x, \quad y \to y, \quad z \to az.$$

Let $F(y,z) := \phi^{-1}(B(y,z))$ and

$$f := x^2 + xz + F(y,z) \quad g := xy + \phi^{-1}(d/a)yz + \alpha y^r + \phi^{-1}(\beta/a^s)z^s.$$

Then we get

$$\phi(f) = x^2 + axz + B(y,z)$$

$$\phi(g) = xy + (d/a)yaz + \alpha y^r + (\beta/a^s)(a^s z^s) = xy + dyz + \alpha y^r + \beta z^s.$$

We remark that the constant term of the power series d/a is 1 and the power series β/a^s is invertible if not zero. Hence the same holds for $\phi^{-1}(d/a)$ and $\phi^{-1}(\beta/a^s)$. The conclusion follows. $\square$

We recall that in this section we are assuming that $A = \mathrm{K}[[x,y,z]]/I$ is a local ring of type $(2,2)$ such that $HF_A(1) = 3$, $HF_A(2) = 4$ and $HF_A(3) = 5$. This implies that if we let n to be **the least integer such that** $HF_A(n) = HF_A(n+1)$, then $n \geq 3$. Also it is easy to see that $n \leq r$, the reduction number of A. The integer n plays a relevant work in the paper. With the aid of this integer n and as a consequence of Proposition 2.2, the Hilbert function of a local ring A of type $(2,2)$ and multiplicity e has the following shape:

$$HF_A(t) = \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, n, \\ t+1 & t = n+1, \cdots, e-1, \\ e & t \geq e. \end{cases} \tag{3}$$

for some integer $n \leq e - 2$. We say that HF_A has a **flat** in position n.

It is clear that we have two possibilities, either $e = n + 2$ or $e \geq n + 3$. In the first case the Hilbert function is increasing by one up to reach the multiplicity, while in the second case the Hilbert function has a flat in position n and is increasing by one in all the other positions, before reaching the multiplicity.

We are ready to prove the main result of this paper. It says that, quite unexpectedly, if the Hilbert function of a local ring of type $(2,2)$ has a flat in position n, then the multiplicity cannot be too big, namely it cannot overcome $2n$.

First we need this easy Lemma.

Lemma 3.5. *Let* $J \subset P = k[x,y,z]$ *be a monomial ideal such that* $x^2, xy \in J$. *If for some* $n \geq 2$ *we have* $HF_{P/J}(n+1) = n+2$ *and* $HF_{P/J}(n+2) = n+3$, *then* xz^n *is the unique monomial of degree* $n+1$ *which is in* J *and not in* (x^2, xy).
If we have also $HF_{P/J}(n) = n+2$, *then* $J_d = (x^2, xy)_d$ *for all* $2 \leq d \leq n$.

Proof. Since $HF_{P/J}(n+1) = n+2 < HF_{P/(x^2,xy)}(n+1) = n+3$ there is a monomial m of degree $n+1$ which is in J and not in (x^2, xy). If $m \neq xz^n$ it should be $m = y^{n+1-j}z^j$ for some $j = 0, ..., n+1$. But then the monomials of the vector space $(x^2, xy)_{n+2}$ and my, mz would be linearly independent. This implies that

$$n+3 = HF_{P/J}(n+2) \leq HF_{P/(x^2,xy,my,mz)}(n+2) = HF_{P/(x^2,xy)}(n+2) - 2 = n+2,$$

a contradiction. Hence $m = xz^n$.

Let us assume that also $HF_{P/J}(n) = n+2$; if for some $t \leq n-1$ we have $HF_{P/J}(t) \leq t+1$, then

$$HF_{P/J}(t+1) \leq HF_{P/J}(t)^{<t>} \leq (t+1)^{<t>} = t+2$$

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and going on in this way we would have $HF_{P/J}(n) \le n+1$, a contradiction. It follows that for all $2 \le d \le n$ we have $HF_{P/J}(d) = HF_{P/(x^2,xy)}(d)$ and, as a consequence, $J_d = (x^2, xy)_d$ for the same d's. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.6. *Let A be a local ring of type (2,2) and multiplicity e. If the Hilbert function of A has a flat in position n, then $e \le 2n$.*

Proof. As usual, we consider a monomial ordering τ on the terms of $K[x, y, z]$ such that $x >_\tau z$. In order to cover both case a) and b) in Theorem 3.4, we may assume $I = (f, g)$ where

$$f := x^2 + axz^p + F(y, z) \qquad g := xy + G(y, z)$$

are power series such that $p \ge 1$ and $a = 0$ or $a \in \mathbb{U}(K[\![z]\!])$.

Since it is clear that $(x^2, xy) \not\subseteq \mathrm{Lt}_\tau(I)$, the elements f and g are not a standard basis for I; thus, by Buchberger's criterion, we should have

$$h := \mathrm{NF}(S(f, g), \{f, g\}) \ne 0.$$

It is clear that $h \in I$ and if we let $m := \mathrm{Lt}_{\overline{\tau}}(h) = \mathrm{Lt}_\tau(\mathrm{h}^*)$, then $m \in \mathrm{Lt}_{\overline{\tau}}(I)$ and by 1.6.4 in [19] $m \notin (x^2, xy)$. We claim that m is a monomial of degree $n+1$.

Namely, by the second statement of Lemma 3.5 applied to the monomial ideal $\mathrm{Lt}_{\overline{\tau}}(I)$, it is clear that m has degree at least $n+1$. Let us assume that m has degree $\ge n+2$, so that $\mathrm{order}(h) \ge n+2$. Since for every s and G one can easily prove that $\mathrm{order}(s) \le \mathrm{order}(\mathrm{NF}(s\,|G))$, we get

$$\mathrm{order}(\mathrm{NF}(S(f, h)|\{f, g, h\})) \ge \mathrm{order}(S(f, h)) \ge max\{\mathrm{order}(f), \mathrm{order}(h)\} \ge n+2.$$

In the same way we can also prove that $\mathrm{order}(\mathrm{NF}(S(g, h)|\{f, g, h\})) \ge n+2$. Now recall that, accordingly to the Buchberger algorithm, in order to determine a standard basis of $\mathrm{Lt}_{\overline{\tau}}(I)$, one has to compute $\mathrm{NF}(S(f, h)|\{f, g, h\})$, $\mathrm{NF}(S(g, h)|\{f, g, h\})$, to add those of them which are not zero to the list and go on in this way up to the end. At each step of this procedure the order of the elements can only increase; hence if m has degree $\ge n+2$ then $\mathrm{NF}(S(f, h)|\{f, g, h\})$ and $\mathrm{NF}(S(g, h)|\{f, g, h\})$ have degree at least $n+2$ and we cannot obtain, as Lemma 3.5 requires, the monomial xz^n which has degree $n+1$. This proves the claim. By Lemma 3.5 the claim implies that $m = \mathrm{Lt}_{\overline{\tau}}(h) = \mathrm{Lt}_\tau(\mathrm{h}^*) = xz^n$.

We want now to compute $\mathrm{NF}(S(f, g) \mid \{f, g\})$. First it is clear that we can write $G(y, z) = yH(y, z) + \alpha z^c$ with $\alpha = 0$ or invertible in $K[\![z]\!]$ and $c \ge 0$. Hence $I = (f, g)$ where

$$f = x^2 + axz^p + F(y, z), \quad g = xy + yH(y, z) + \alpha z^c$$

with $c \ge 0$, $p \ge 1$ and a and α either zero or invertible in $K[\![z]\!]$. We have

$$S(f, g) = yf - xg = axyz^p + yF(y, z) - xyH(y, z) - \alpha xz^c = g(az^p - H(y, z)) - \alpha xz^c + M(y, z)$$

where $M(y, z) = yF(y, z) - (yH(y, z) + \alpha z^c)(az^p - H(y, z)) \in K[\![y, z]\!]$.

We claim that $\mathrm{NF}(S(f, g) \mid \{f, g\}) = M(y, z) - \alpha xz^c$. Namely we have

$$S(f, g) = 0 \cdot f + (az^p - H(y, z))g + M(y, z) - \alpha xz^c$$

and we need to prove:

a) no monomial in the support of $M(y, z) - \alpha xz^c$ is divisible by x^2 or xy

b) $\mathrm{Lt}_{\overline{\tau}}(g(az^p - H(y, z))) \le \mathrm{Lt}_{\overline{\tau}}(S(f, g))$.

Now a) is true because α is zero or invertible in $K[\![z]\!]$. As for b) it is clear that we have $\mathrm{Lt}_{\overline{\tau}}(g(az^p - H(y, z))) = xy \cdot \mathrm{Lt}_{\overline{\tau}}(az^p - H(y, z))$. This monomial is not in the support of $M(y, z) - \alpha xz^c$, hence it is in the support of $S(f, g)$. This implies b) and the claim

$$h = \mathrm{NF}(S(f, g) \mid \{f, g\}) = M(y, z) - \alpha xz^c \tag{4}$$

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is proved.

Since $\mathrm{Lt}_{\tau}(h) = xz^n$ it follows $\alpha \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $c = n$, and $\mathrm{order}(M) \geq n+1$. In particular we deduce

$$g = xy + \alpha z^n + yH(y, z). \qquad (5)$$

Let $J := I + (y) = (x^2 + axz^p + F(0, z), z^n, y)$; it is clear that $\mathrm{Lt}_{\tau}(J) \supseteq (x^2, z^n, y)$. Since R/J is Artinian, $\overline{y}$ is a parameter in $A = R/I$; hence

$$e = e(R/I) \leq length(R/J) = length(R/\mathrm{Lt}_{\tau}(J) \leq length(R/(x^2, z^n, y)) = 2n.$$

The conclusion follows. $\qquad\square$

In example 2.3, we have seen that however we fix an integer $e \geq 4$, there is a local ring of type $(2, 2)$ with multiplicity e and strictly increasing Hilbert function. For each pair of integers (n, e) such that $n \geq 3$ and $n + 3 \leq e \leq 2n$, we exhibit now local rings of type $(2, 2)$ and multiplicity e whose Hilbert function has a flat in position n.

Example 3.7. Given the integers n and e such that $n \geq 3$, $n + 3 \leq e \leq 2n$, the ideal

$$I = (x^2 - y^{e-2}, xy - z^n)$$

is a complete intersection ideal of $R = \mathrm{K}[\![x, y, z]\!]$ of type $(2, 2)$ with multiplicity e, whose Hilbert function has a flat in position n.

Proof. Let us consider a monomial ordering τ such that $x > y > z$; we are going to prove that

$$\{f = x^2 - y^{e-2}, \quad g = xy - z^n, \quad h = -y^{e-1} + xz^n, \quad k = y^e - z^{2n}\}$$

is a standard basis for I. Namely, if this is the case, we get $\mathrm{Lt}_{\tau}(I) = (x^2, xy, xz^n, y^e)$ and from this an easy computation shows that the local ring $K[x, y, z]]/(x^2 - y^{e-2}, xy - z^n)$ has multiplicity e and Hilbert function with a flat in position n.

We have:
$$S(f, g) = yf - xg = y(x^2 - y^{e-2}) - x(xy - z^n) = xz^n - y^{e-1}$$

and since $e \geq n + 3$ implies $e - 1 \geq n + 2 > n + 1$, we get $\mathrm{Lt}_{\tau}(S(f, g)) = xz^n$.

We let
$$h := S(f, g) = xz^n - y^{e-1}.$$

Now
$$S(f, h) = z^n f - xh = z^n(x^2 - y^{e-2}) - x(xz^n - y^{e-1}) = xy^{e-1} - z^n y^{e-2} = y^{e-2}g$$

so that $\mathrm{Lt}_{\tau}(S(f, h)) = y^{e-2}\mathrm{Lt}_{\tau}(g) = xy^{e-1}$.

Further
$$S(g, h) = z^n g - yh = z^n(xy - z^n) - y(xz^n - y^{e-1}) = y^e - z^{2n}$$

and since $e \leq 2n$ and $y > z$, we have $\mathrm{Lt}_{\tau}(S(g, h)) = y^e$.

We let
$$k := S(g, h) = y^e - z^{2n}$$

with $\mathrm{Lt}_{\tau}(S(g, h)) = \mathrm{Lt}_{\tau}(k) = y^e$.

Now
$$S(f, k) = y^e f - x^2 k = y^e(x^2 - y^{e-2}) - x^2(y^2 - z^{2n}) = x^2 z^{2n} - y^{2e-2} = z^{2n} f - y^{e-2} k$$

and since $2e - 2 \geq 2(n + 3) - 2 = 2n + 4 > 2n + 2$, we have $\mathrm{Lt}_{\tau}(S(f, k)) = x^2 z^{2n}$.

Also
$$S(g, k) = y^{e-1}g - xk = y^{e-1}(xy - z^n) - x(y^2 - z^{2n}) = xz^{2n} - y^{e-1}z^n = z^n h$$

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so that $\mathrm{Lt}_{\overline{\tau}}(S(g,k)) = z^n \mathrm{Lt}_{\overline{\tau}}(h) = xz^{2n}$.

Finally

$$S(h,k) = y^2 h - xz^n k = y^e(xz^n - y^{e-1}) - xz^n(y^e - z^{2n}) = xz^{3n} - y^{2e-1} = z^{2n}h - y^{e-1}k.$$

Here we can only remark that $\mathrm{Lt}_{\overline{\tau}}(S(h,k)) = \max\{xz^{3n}, y^{2e-1}\}$.

From these computations we get

$$\mathrm{NF}(S(f,g) \mid \{h\}) = \mathrm{NF}(h \mid \{h\}) = 0$$

$$\mathrm{NF}(S(f,h) \mid \{g\}) = \mathrm{NF}(y^{e-2}g \mid \{g\}) = 0$$

$$\mathrm{NF}(S(g,h) \mid \{k\}) = \mathrm{NF}(k \mid \{k\}) = 0$$

$$\mathrm{NF}(S(f,k) \mid \{f,k\}) = \mathrm{NF}(z^{2n}f - y^{e-2}k \mid \{f,k\}) = 0$$

because $\mathrm{Lt}_{\overline{\tau}}(z^{2n}f - y^{e-2}k) = x^2 z^{2n} \geq \mathrm{Lt}_{\overline{\tau}}(z^{2n}f) = x^2 z^{2n}, \mathrm{Lt}_{\overline{\tau}}(y^{e-2}k) = y^{2e-2}$.

$$\mathrm{NF}(S(g,k) \mid \{h\}) = \mathrm{NF}(z^n h \mid \{h\}) = 0$$

$$\mathrm{NF}(S(h,k) \mid \{h,k\}) = \mathrm{NF}(z^{2n}h - y^{e-1}k \mid \{h,k\}) = 0$$

because $\mathrm{Lt}_{\overline{\tau}}(z^{2n}h - y^{e-1}k) = \max\{xz^{3n}, y^{2e-1}\} \geq \mathrm{Lt}_{\overline{\tau}}(z^{2n}h) = xz^{3n}, \mathrm{Lt}_{\overline{\tau}}(y^{e-1}k) = y^{2e-1}$.

By Buchberger's criterion the conclusion follows. $\qquad\square$

We prove now that if I_2^* is square-free then the Hilbert function is strictly increasing, so that the associated graded ring is Cohen-Macaulay.

Proposition 3.8. *Let $A = R/I$ be a local ring of type $(2,2)$. If I^*_2 is square-free, then the Hilbert function of A is strictly increasing and thus the associated graded ring $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay.*

Proof. From Proposition 3.4 (ii) we may assume, up to isomorphism of R, that

$$I = (x^2 + xz + F(y,z), xy + byz + \alpha y^r + \beta z^s)$$

where $F \in K[\![y,z]\!]_{\geq 3}$, $b \in K[\![y,z]\!]$ with $b(0,0) = 1$, $r, s \geq 3$, $\alpha = 0$ or $\alpha \in \mathbb{U}(K[\![y]\!])$, and $\beta = 0$ or $\beta \in \mathbb{U}(K[\![z]\!])$.

If $HF_{R/I}(n+2) = n+2$, then $e = n+2$ and we conclude by Proposition 2.5. Assume that $HF_{R/I}(n+2) = n+3$, then by Lemma 3.5 we have that $xz^n \in \mathrm{Lt}_{\overline{\tau}}(I)$. By Burchberger's criterion we should have $xz^n = \mathrm{Lt}_{\overline{\tau}}(\mathrm{NF}(S(f,g), \{f,g\}))$. The S-polynomial of the pair f, g is

$$h := S(f,g) = z(b-1)A + \alpha y^{r-1}A + yF - \beta xz^s,$$

$A = byz + \alpha y^r + \beta z^s$. We write $L = z(b-1)A + \alpha y^{r-1}A + yF$; notice that $L \in K[\![y,z]\!]$ and $\beta \in K[\![z]\!]$ so $h = \mathrm{NF}(h, \{f,g\})$. Since $\mathrm{Lt}_{\overline{\tau}}(h) = xz^n$ we deduce $\beta \in \mathbb{U}(K[\![z]\!])$, $s = n$, and $\mathrm{order}(L) \geq n+1$. Let now consider

$$\begin{aligned}
S(h,g) = W &= \beta z^n g + yh \\
&= \beta byz^{n+1} + \alpha\beta y^r z^n + \beta^2 z^{2n} + yL.
\end{aligned}$$

Notice that since $b, \beta \neq 0$

$$\mathrm{order}(\alpha\beta y^r z^n), \mathrm{order}(\beta^2 z^{2n}) \geq n+3 > n+2 = \mathrm{order}(\beta byz^{n+1})$$

and $\mathrm{Lt}_{\overline{\tau}}(\beta byz^{n+1}) = yz^{n+1}$.

Recall that $\mathrm{order}(L) \geq n+1$, so in order to prove that $\mathrm{Lt}_{\overline{\tau}}(W) = yz^{n+1}$ we should prove that in $\mathrm{Supp}(yL)$ there is not the monomial yz^{n+1}. This is equivalent to prove that in $\mathrm{Supp}(L)$ there is not the monomial z^{n+1}. At this end we set $y = 0$ in L and we get

$$L(0,z) = (b(0,z) - 1)\beta z^{n+1}.$$

recall that $b(0,0) = 1$ so order$(L(0,z)) \geq n+2$. Hence we have that $\text{Lt}_{\overline{\tau}}(k) = yz^{n+1}$.

Let us consider now the monomial ideal $J = (x^2, xy, xz^n, yz^{n+1}) \subset \text{Lt}_{\overline{\tau}}(I)$. We have

$$HF_{R/I}(n+2) \leq HF_{R/J}(n+2) = n+2,$$

a contradiction. $\qquad\square$

Notice that if I_2^* is square-free then by Lemma 3.2 (ii) we may assume, up to isomorphisms, that $f^* = xy, g^* = xz$. Hence Proposition 3.8 recover [7, Corollary 4.6] in the case $(2,2)$.

The following example shows that the converse of the above theorem does not hold. Let $I = (x^2 - y^2z, xy - y^3) \subseteq \text{K}[\![x,y,z]\!]$. It is clear that $x^2 \in I_2^*$ so that I_2^* is not square-free. It is easy to see that the Hilbert function of $A = \text{K}[\![x,y,z]\!]/I$ is strictly increasing, namely is $\{1,3,4,5,5,5,5,.......\}$. By Proposition 2.5 the associated graded ring of A is Cohen-Macaulay.

We close this section by describing the possible minimal free resolutions of the associated graded ring of a local ring of type $(2,2)$.

We have seen in (3) that the Hilbert function of a local ring A of type $(2,2)$ has the following shape

$$HF_A(t) = \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, n, \\ t+1 & t = n+1, \cdots, e-1, \\ e & t \geq e. \end{cases} \qquad (6)$$

where n is the least integer such that $HF_A(n) = HF_A(n+1)$. We have $3 \leq n \leq e-2$ and it is easy to see that the lex-segment ideal with the above Hilbert function is the following ideal $L := (x^2, xy, xz^n, y^e)$. We can compute the minimal free resolution of P/L by using the well known formula of Eliaouh and Kervaire. We get

$$0 \to P(-n-3) \to P(-3) \oplus P(-n-2)^2 \oplus P(-e-1) \to$$
$$\to P(-2)^2 \oplus P(-n-1) \oplus P(-e) \to P \to P/L \to 0.$$

It is clear that in the case $e \geq n+3$ there is no possible cancelation. Hence every homogeneous ideal J with Hilbert function as in (3) and with $e \geq n+3$ has the same resolution of the corresponding lex-segment ideal.

In the other case, when $e = n+2$, we can either have the above resolution or one of the following obtained by cancelation:

$$0 \to P(-n-3) \to P(-3) \oplus P(-n-2) \oplus P(-n-3) \to P(-2)^2 \oplus P(-n-1) \to P \to P/J \to 0$$

$$0 \to P(-3) \oplus P(-n-2) \to P(-2)^2 \oplus P(-n-1) \to P \to P/J \to 0.$$

It is clear that if P/J is Cohen-Macaulay only the last shorter resolution is available.

We apply this to the associated graded ring of a local ring of type $(2,2)$ and we get the following result.

Proposition 3.9. *Let A be a local ring of type ($2,2$), e the multiplicity of A and let n be an integer such that $n \leq e-2$. If $e = n+2$, then $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay with minimal free resolution:*

$$0 \to P(-3) \oplus P(-n-2) \to P(-2)^2 \oplus P(-n-1) \to P \to gr_{\mathfrak{m}}(A) \to 0.$$

If $e \geq n+3$, then $e \leq 2n$ and $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay with minimal free resolution

$$0 \to P(-n-3) \to P(-3) \oplus P(-n-2)^2 \oplus P(-e-1) \to$$
$$\to P(-2)^2 \oplus P(-n-1) \oplus P(-e) \to P \to gr_{\mathfrak{m}}(A) \to 0.$$

Proof. It is enough to remark that by Proposition 2.5 the associated graded ring of a local ring of type $(2,2)$ is Cohen-Macaulay when $e = n+2$. $\qquad\square$

4 A structure's theorem for quadratic complete intersections of codimension two

The aim of this section is to give a structure, up analytic isomorphism, of the minimal system of generators of ideals I of type $(2,2)$ such that $A = K[\![x,y,z]\!]/I$ is of multiplicity e. This a first step towards the difficult problem of the analytic classification of the ideals of type $(2,2)$. In this direction we show in Example 5.8 two ideals of type $(2,2)$ with same Hilbert function that are not analytic isomorphic. Accordingly with Proposition 2.2, Example 2.3, and Example 3.7 the Hilbert function of A take exactly the following shapes

$$H(e)(t) := \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, e-3, \\ e & t \geq e-2 \end{cases} \tag{7}$$

or

$$H(n,e)(t) := \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, n, \\ t+1 & t = n+1, \cdots, e-2, \\ e & t \geq e-1. \end{cases} \tag{8}$$

Theorem 4.1. *Let A be a local ring of type $(2,2)$ and multiplicity e. The following conditions are equivalent:*
(i) $HF_A = H(n,e)$ for some integer $n \geq 3$.
(ii) Up analytic isomorphism, I is generated in $R = K[\![x,y,z]\!]$ by:

$$\begin{aligned} f &= x^2 + az^p(x+H) - H^2 + L \\ g &= xy + \alpha z^n + yH \end{aligned}$$

where

- $a \in \{0,1\}$, $p \geq 2$, $\alpha \in \mathbb{U}(K[\![z]\!])$,

- $H, L \in K[\![y,z]\!]$ with $\mathrm{order}(L) \geq n+1$, $\mathrm{order}(H) \geq 2$,

- $n+3 \leq e \leq 2n$,

- $\mathrm{order}(2\alpha z^n H - a\alpha z^{n+p} + yL) \geq e-1$ *and the equality holds whenever* $e < 2n$.

Proof. Taking advantage of Proposition 3.4, the proof is based on the computation of $\mathrm{Lt}_{\overline{\tau}}(I)$ accordingly with Buchberger's criterion. As usual assume $x > y, x > z$.
First we prove (i) implies (ii). Since $HF_A = H(n,e)$, then by Proposition 2.5 $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay and, by Theorem 3.6, $n+3 \leq e \leq 2n$. By Proposition 3.8, I_2^* contains a square of a linear form. Hence we may assume that $f^* = x^2$ and $g^* = xy$. Notice that $x^2, xy \in \mathrm{Lt}_{\overline{\tau}}(I)$, hence because (i), by Lemma 3.5, $\mathrm{Lt}_{\overline{\tau}}(I) \supseteq (x^2, xy, xz^n)$. From the particular shape of the Hilbert function it is easy to see that $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, xz^n, m)$ where m is a monomial in $K[y,z]_e$. From Lemma 3.4 we may also assume

$$\begin{aligned} f &= x^2 + axz^p + F(y,z), \\ g &= xy + G(y,z), \end{aligned}$$

where $a \in \{0,1\}$, $p \geq 2$, $F, G \in K[\![y,z]\!]$ with $\mathrm{order}(F), \mathrm{order}(G) \geq 3$. Moreover, from the equation (5) of the proof of Theorem 3.6, (5), we get

$$G(y,z) = yH(y,z) + \alpha z^n$$

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where $H \in \mathrm{K}[\![y,z]\!]_{\geq 2}$ and $\alpha \in \mathbb{U}(\mathrm{K}[\![z]\!])$. We recall that $S(f,g) = yf - xg = axyz^p + yF - \alpha xz^n - xyH$. In particular a standard computation gives

$$h := \mathrm{NF}(S(f,g),\{f,g\}) = -\alpha xz^n + yL + \alpha z^n(H - az^p)$$

where $L = F - az^pH + H^2$. Notice that $\mathrm{order}(\alpha z^n(H - az^p)) \geq n+2$. Notice that $xz^n = \mathrm{Lt}_{\overline{\tau}}(h)$.

A simple calculation shows that $\mathrm{NF}(S(h,f),\{h,f,g\}) = 0$. On the other hand

$$S(h,g) = \mathrm{NF}(S(h,g),\{h,f,g\}) = \alpha^2 z^{2n} + y(2\alpha z^n H - a\alpha z^{n+p} + yL) \neq 0$$

because $\alpha \neq 0$ and z^{2n} does not appear in the support of the remaining part. As a consequence $m = \mathrm{Lt}_{\overline{\tau}}(S(h,g))$, and hence $\mathrm{order}(S(h,g)) = e$. It follows $\mathrm{order}(2\alpha z^n H - a\alpha z^{n+p} + yL) \geq e-1$. In particular $\mathrm{order}(L) \geq n+1$, and $\mathrm{order}(2\alpha z^n H - a\alpha z^{n+p} + yL) = e-1$ if $e < 2n$.

Conversely, assuming (ii), it is enough to apply Buchberger's criterion for computing $\mathrm{Lt}_{\overline{\tau}}(I)$. By following the previous computations we get $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, xz^n, m)$ where $m = \mathrm{Lt}_{\overline{\tau}}(\alpha^2 z^{2n} + y(2\alpha z^n H - a\alpha z^{n+p} + yL))$ and (i) follows. $\qquad\square$

If the Hilbert function is increasing, i.e. of type $H(e)$, we present a structure's theorem under the assumption that I^* does not contain the square of a linear form.

Theorem 4.2. *Let A be a local ring of type $(2,2)$ and multiplicity e. The following conditions are equivalent:*
(i) $HF_A = H(e)$ and I^* does not contain the square of a linear form
(ii) Up analytic isomorphism, I is generated in $R = \mathrm{K}[\![x,y,z]\!]$ by:

$$\begin{aligned} f &= x^2 + xz + F \\ g &= xy + dyz + \alpha y^r + \beta z^s \end{aligned}$$

where

- $r \geq 3$,

- $F \in \mathrm{K}[\![y,z]\!]$ *and* $\mathrm{order}(F) \geq 3$,

- $d \in \mathbb{U}(\mathrm{K}[\![y,z]\!])$, *with* $d(0,0) = 1$,

- $\alpha = 0$ *or* $\alpha \in \mathbb{U}(\mathrm{K}[\![y]\!])$, $\beta = 0$ *or* $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$ *and* $s \geq e-1 \geq 3$,

- $\mathrm{order}(F + d(d-1)z^2 + \alpha(2d-1)zy^{r-1} + \alpha^2 y^{2(r-1)}) = e-2$.

Proof. As usual, consider a monomial ordering τ with $x > y, x > z$. We prove (i) implies (ii). By Theorem 3.4 (ii), we may assume that

$$I = (x^2 + xz + F(y,z), xy + dyz + \alpha y^r + \beta z^s)$$

$F \in \mathrm{K}[\![y,z]\!]_{\geq 3}$, $d \in \mathrm{K}[\![y,z]\!]$ with $d(0,0) = 1$, $r,s \geq 3$, $\alpha = 0$ or $\alpha \in \mathbb{U}(\mathrm{K}[\![y]\!])$, and $\beta = 0$ or $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$. Since the Hilbert function is increasing up to $n = e-2 \geq 2$ and $HF_{R/I}(t) = e$ for all $t \geq e-2$, then $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, m)$ where $m \in K[y,z]$ is a monomial of degree $e-1$.

By Buchberger's criterion necessarily $m = \mathrm{Lt}_{\overline{\tau}}(\mathrm{NF}(S(f,g),\{f,g\}))$. Now

$$S(f,g) = -\beta xz^s + yF(y,z) + xy[(1-d)z - \alpha y^{r-1}]$$

After a computation we get

$$\mathrm{NF}(S(f,g),\{f,g\}) = -\beta xz^s + yW + \alpha\beta y^{r-1}z^s + \beta(d-1)z^{s+1}$$

where $W = F + d(d-1)z^2 + \alpha(2d-1)zy^{r-1} + \alpha^2 y^{2(r-1)}$. Since $yW \in \mathrm{K}[\![y,z]\!]$, $r \geq 3$ and $1 - d \in (y,z)\mathrm{K}[\![y,z]\!]$ we get that if $\beta \neq 0$, then xz^s appears in the support of $\mathrm{NF}(S(f,g),\{f,g\})$. Since $\mathrm{Lt}_{\overline{\tau}}(\mathrm{NF}(S(f,g),\{f,g\})) \in K[y,z]_{e-1}$, it follows $\mathrm{order}(W) = e-2$ and, if $\beta \neq 0$, then $s \geq e-1$.

Conversely if we assume (ii), then it is easy to see that I^* does not contain the square of a linear form because $I_2^* = (x^2 + xz, xy + yz)$ which is reduced. Moreover by repeating Buchberger's algorithm, looking at the previous computation on $S(f, g)$, we get

$$\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, y\mathrm{Lt}_{\overline{\tau}}(W)),$$

hence $HF_A = H(e)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5 Examples

The aim of this section is to present examples supporting the results of the previous sections or detecting the possible extensions to the non quadratic case. All computations are performed by using CoCoA system ([2]). Here $HS_A(\theta)$ denotes the Hilbert series of A, that is $HS_A(\theta) = \sum_{t \geq 0} HF_A(t)\theta^t$.

We have seen in Proposition 3.9 that the minimal free resolution of the tangent cone of a local ring of type $(2, 2)$ has no possible cancelation, both in the case the Hilbert function is strictly increasing and in the case of a flat. One can ask if this is the case also for local rings of type (a, b) with $3 \leq a \leq b$.

The first two examples that we propose show that the answer is negative.

Example 5.1. Let $A = R/I$ where $I = (x^3, z^5 + xz^3 + x^2y)$. The local ring A has type $(3, 3)$ and $I^* = (x^3, x^2y, x^2z^3, -xyz^5 + xz^6, -xz^7, z^{10})$. The resolution of P/I^* is the following

$$0 \to P(-7) \oplus P(-10) \to P(-4) \oplus P^2(-6) \oplus P(-8) \oplus P^2(-9) \oplus P(-11) \to$$

$$\to P^2(-3) \oplus P(-5) \oplus P(-7) \oplus P(-8) \oplus P(-10) \to P \to P/I^* \to 0.$$

It is clear that we have a possible cancelation and the Hilbert function

$$\{1, 3, 6, 8, 10, 11, 13, 14, 14, 15, 15, ...\}$$

has a flat in position 7.

Example 5.2. Let $A = R/I$ where $I = (x^4, z^4 + x^2y)$. The local ring A has type $(3, 4)$ and $I^* = (x^2y, x^4, x^2z^4, z^8)$. The resolution of P/I^* is the following

$$0 \to P(-9) \to P(-5) \oplus P(-7) \oplus P(-8) \oplus P(-10) \to$$

$$\to P(-3) \oplus P(-4) \oplus P(-6) \oplus P(-8) \to P \to P/I^* \to 0.$$

It is clear that we have a possible cancelation and the Hilbert function

$$\{1, 3, 6, 9, 11, 13, 14, 15, 16, 16, 16, ...\}$$

is strictly increasing.

The following example shows that Proposition 2.5 cannot be extended to local rings of type (a, b) with $a > 2$.

Example 5.3. Let us consider the ideal $I = (x^4, x^2y + z^4) \subseteq R = k[[x, y, z]]$. Then $A = R/I$ has strictly increasing Hilbert function, in fact the Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 3\theta^2 + 3\theta^3 + 2\theta^4 + 2\theta^5 + \theta^6 + \theta^7 + \theta^8)/(1 - \theta).$$

Nevertheless $I^* = (x^2y, x^4, x^2z^4, z^8)$, hence $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay.

The following example, due to T. Shibuta, shows that the Hilbert function of a one-dimensional local domain of type $(2, b)$ can have $b - 1$ flats, the maximum number accordingly to Proposition 2.

Example 5.4. (see [7], example 5.5) Let $b \geq 2$ be an integer. Consider the family of semigroup rings

$$A = k[[t^{3b}, t^{3b+1}, t^{6b+3}]].$$

It is easy to see that $A = k[[x, y, z]]/I_b$ where $I_b = (xz - y^3, z^b - x^{2b+1})$. Thus A is a one-dimensional local domain of type $(2, b)$. For every $b \geq 2$ the Hilbert function of A has $b - 1$ flats. Namely

$$HF_A(t) = \begin{cases} 1 & t = 0, \\ 2t + 2 & t = 1, \cdots, b - 1, \\ 2b & t = b, \\ 2b + 1 & t = b + 1, \\ 2b + k & t = b + 2k, \quad k = 1, \cdots, b - 1, \\ 2b + k + 1 & t = b + 2k + 1, \quad k = 1, \cdots, b - 1, \\ 3b & t \geq 3b - 1. \end{cases} \tag{9}$$

In the above example the Hilbert function of the local ring of type $(2, b)$ presents $b - 1$ flats which are not consecutive. The following example shows that we can also have $b - 1$ consecutive flats, that is a strip like this: $HF(n) = HF(n + 1) = \cdots = HF(n + b - 1) < e$.

Example 5.5. Let us consider the ideal $I = (x^2, xy^2 + z^5 + xy^3z^2) \subseteq R = k[[x, y, z]]$. Then $A = R/I$ is of type $(2, 3)$ and its Hilbert function presents two (=b-1) flats which are consecutive: namely we have $HF(5) = HF(6) = HF(7) = 8 < e = 10$. In particular the Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 2\theta^2 + \theta^3 + \theta^4 + \theta^5 + \theta^8 + \theta^9)/(1 - \theta).$$

The Hilbert function of a local ring of type (a, b), with $3 \leq a \leq b$, is at the moment far from our understanding. In order to show how the problem is difficult when a and b are increasing, we present two more examples, the first of type $(3, 3)$ with one very large platform consisting of 13 consecutive flats, the second of type $(4, 4)$ with nine flats and three platforms.

Example 5.6. Let $I = (x^3 - zy^{14}, x^2y + xz^7) \subseteq R = k[[x, y, z]]$. The local ring $A = R/I$ is of type $(3, 3)$ and

$$HF_A(15) = HF_A(16) = \ldots \cdots = HF_A(29) = 31 < e = 32.$$

In particular the Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 3\theta^2 + 2\theta^3 + 2\theta^4 + 2\theta^5 + 2\theta^6 + 2\theta^7 + 2\theta^8 + \theta^9 + 2x\theta^{10} + 2\theta^{11} + 2\theta^{12} +$$

$$+2\theta^{13} + 2\theta^{14} + \theta^{15} + \theta^{16} + \theta^{30} + \theta^{31})/(1 - \theta)$$

and $I^* = (x^3, x^2y, x^2z^7, xz^{14}, xy^{15}z, y^{31}z)$.

Example 5.7. Let $I = (x^4, xy^3 - z^6) \subseteq R = k[[x, y, z]]$. The local ring $A = R/I$ is of type $(4, 4)$ and

$$HF_A(8) = HF_A(9) = HF_A(10) = HF_A(11) = 18;$$

$$HF_A(13) = HF_A(14) = HF_A(15) = HF_A(16) = 20;$$

$$HF_A(18) = HF_A(19) = HF_A(20) = HF_A(21) = 22 < e = 24;$$

In particular the Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 3\theta^2 + 4\theta^3 + 3\theta^4 + 2\theta^5 + \theta^6 + \theta^7 + \theta^8 + \theta^{12} + \theta^{13} + \theta^{17} + \theta^{18} + \theta^{22} + \theta^{23})/(1 - \theta)$$

and $I^* = (xy^3, x^4, x^3z^6, x^2z^{12}, xz^{18}, z^{24})$.

It would be very interesting to describe the isomorphism classes of local rings of type $(2, 2)$ which have the same given Hilbert function. But this is a difficult task, as the following examples show.

First we are given the Hilbert function $\{1, 3, 4, 5, 5, 6, 6, ...\}$ which has a flat in position 3 and multiplicity 6. The two ideals which we are going to prove that are not isomorphic are obtained one from the other with very little modifications, namely by adding a monomial to one of the two generators.

Example 5.8. Let us consider the ideals

$$I = (x^2 - y^4, xy + z^3), \quad J = (x^2 + xz^2 - y^4, xy + z^3)$$

in $R = K[\![x, y, z]\!]$.

They are of type $(2, 2)$, they have the same Hilbert function $\{1, 3, 4, 5, 5, 6, 6, ...\}$ and the same leading ideal $\mathrm{Lt}_{\overline{\tau}}(I) = \mathrm{Lt}_{\overline{\tau}}(J) = (x^2, xy, xz^3, y^6)$. On the other hand the ideals of initial forms differ in degree 6:

$$I^* = (x^2, xy, xz^3, y^6 - z^6), \quad J^* = (x^2, xy, xz^3, y^6 + yz^5 - z^6).$$

We prove that $K[\![x, y, z]\!]/I$ and $K[\![x, y, z]\!]/J$ are not isomorphic.

If there exists an analytic isomorphism ϕ such that $\phi(I) = J$ then we can find power series f, g, h of order 1 such that $\mathcal{M} = (f, g, h)$ and ϕ is the result of substituting f for x, g for y and h for z in any power series of R. We have $f = L_1 + F$ $\quad g = L_1 + G$ $\quad h = L_3 + H$ where L_1, L_2, L_3 are linearly independent linear forms in $K[x, y, z]$ and F, G, H are power series of order ≥ 2.

We let for $i = 1, 2, 3$

$$L_i = \lambda_{i1} x + \lambda_{i2} y + \lambda_{i3} z$$

with $\lambda_{ij} \in K$. Since $x^2 - y^4 \in I$ we have $\phi(x^2 - y^4) = f^2 - g^4 \in J$, hence $L_1^2 \in J^*$. Since I^*_2 is the K-vector space $I^*_2 = <x^2, xy>$, we have

$$(\lambda_{11} x + \lambda_{12} y + \lambda_{13} z)^2 = px^2 + qxy$$

with $p, q \in K$; this clearly implies $\lambda_{12} = \lambda_{13} = 0$.

In the same way, since $xy + z^3 \in I$, we have $\phi(xy + z^3) = fg + h^3 \in J$, hence $L_1 L_2 \in J^*$. Thus we get

$$(\lambda_{11} x)(\lambda_{21} x + \lambda_{22} y + \lambda_{23} z) = rx^2 + sxy$$

with $r, s \in K$. This implies $\lambda_{23} = 0$ because $\lambda_{11} \neq 0$.

Finally we have

$$y^6 - z^6 = -y^2(x^2 - y^4) + (xy + z^3)(xy - z^3) \in I$$

so that $\phi(y^6 - z^6) = g^6 - h^6 \in J$, and, as before, $L_2^6 - L_3^6 \in J^*$. Looking at the generators of the vector space J^*_6 we get as a consequence

$$(\lambda_{21} x + \lambda_{22} y)^6 - (\lambda_{31} x + \lambda_{32} y + \lambda_{33} z)^6 = Ax^2 + Bxy + Cxz^3 + D(y^6 + yz^5 - z^6)$$

where A, B, C, D are forms of degree $4, 4, 2, 0$ respectively in the polynomial ring $K[x, y, z]$.

Since L_1, L_2, L_3 are linearly independent, we must have $\lambda_{33} \neq 0$. Hence, looking at the coefficient of the monomial $y^5 z$ in the above formula, we get $\lambda_{32} = 0$. But then, looking at the coefficient of the monomial yz^5, we certainly get $D = 0$ and finally, looking at the coefficient of the monomial z^6, we get $\lambda_{33} = 0$. This is a contradiction, so that the algebras R/I and R/J are not in the same isomorphism class.

The case when the Hilbert function is strictly increasing is not more easy to handle. Here we consider the Hilbert function $\{1, 3, 4, 5, 6, 6, 6, ...\}$ which is strictly increasing and we look at the possible isomorphism classes of local rings with that Hilbert function.

Example 5.9. Let us consider the two ideals

$$I := (x^2 + y^4, xy), \qquad J := (x^2 + y^4 + z^4, xy).$$

They have the same Hilbert function $\{1, 3, 4, 5, 6, 6, 6, ...\}$ and different tangent cones, namely

$$I^* = (x^2, xy, y^5) \qquad J^* = (x^2, xy, y^5 + yz^4).$$

A calculation as before shows that $K[\![x, y, z]\!]/I$ and $K[\![x, y, z]\!]/J$ are not isomorphic.

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